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On piezoelastic contact problem for a smooth punch

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Abstract

This paper firstly conducts a systematic three-dimensional investigation of the problem of a rigid smooth punch bonded to a transversely isotropic piezoelectric half-space. The potential theory method is employed and generalized to take into account the effect of the electric field. In contrast to pure elasticity, two potentials are introduced. For an arbitrarily shaped punch, two governing equations are derived, which can be solved using numerical methods. Particularly, a closed-form, exact solution is obtained for a flat centrally loaded circular punch which is maintained at a constant electric potential. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Piezoelectric medium; Transverse isotropy; Potential theory; Circular punch

1. Introduction

In view of the particular coupling effect, piezoelectric materials have gained a lot of interest from researchers of mechanics. Many theoretical works have been published in the past several decades and some recent developments can be found in Ding et al. (1996, 1997a, b), Sosa and Khutoryansky (1996), Kogan et al. (1996), Huang (1997) and Heyliger (1997), to name a few.

Microhardness testing has been developed as an effective technique to assess mechanical properties of brittle materials such as ceramics, and it allows the introduction of controlled flaws or cracks for strength and fracture toughness evaluation (Lawn and Wilshaw, 1975). Little effort has been made to analyze the contact problems of punches or indenters pressed to piezoelectric materials, which are obviously associated with the above mentioned technology. To the author's knowledge, only Fan et al. (1996) recently considered the two-dimensional contact problem of a piezoelectric half-plane: the non-slip and slip indentor contacts on the half-plane were formulated by the means of Stroh's formalism.

This paper intends to analyze the punch problem of a transversely isotropic piezoelectric half-space completely based on three-dimensional piezoelasticity. To this end, the potential theory method, which has been rigorously developed by Fabrikant (1989) to analyze various mixed boundary value problems in pure elasticity, is employed. The great advantage of this method is that complete and closed-form

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solutions to some classical elastic problems can be derived. This is also the case in piezoelasticity as will be illustrated in the paper by considering a circular punch indenting a piezoelectric half-space.

2. Basic formulations

In Cartesian coordinates (with the z-axis being normal to the plane of isotropy), the linear constitutive relations of a transversely isotropic piezoelectric medium (class 6 mm) are (Tiersten, 1969):

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z}, \quad \tau_{xz} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \Phi}{\partial x},$$

$$\sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z}, \quad \tau_{yz} = c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + e_{15}\frac{\partial \Phi}{\partial y},$$

$$\sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{13}\frac{\partial v}{\partial y} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \Phi}{\partial z}, \quad \tau_{xy} = c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

$$D_x = e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial x}, \quad D_y = e_{15}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial y},$$

$$D_z = e_{31}\frac{\partial u}{\partial x} + e_{31}\frac{\partial v}{\partial y} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z}, \tag{1}$$

where, Φ and D_i are the electric potential and electric displacement vector, respectively; c_{ij}, ε_{ij} and e_{ij} are the elastic, dielectric, and piezoelectric constants, respectively. The conventional notations of stresses and displacements have been employed in eqn (1). It is also noted that there are only five independent elastic constants for transversely isotropic materials, i.e. $c_{11} = c_{12} + 2c_{66}$. The governing equations can be found, for example, in Ding et al. (1996). By introducing the tangential complex displacement $U = u + \mathrm{i}v$, these equations can be rewritten in a complex manner as follows,

$$\frac{1}{2}(c_{11} + c_{66})\Delta U + c_{44}\frac{\partial^2 U}{\partial z^2} + \frac{1}{2}(c_{11} - c_{66})\Lambda^2 \bar{U} + (c_{13} + c_{44})\Lambda\frac{\partial w}{\partial z} + (e_{15} + e_{31})\Lambda\frac{\partial \Phi}{\partial z} = 0,$$

$$\frac{1}{2}(c_{13} + c_{44})\frac{\partial}{\partial z}\left(\bar{\Lambda} U + \Lambda \bar{U}\right) + c_{44}\Delta w + c_{33}\frac{\partial^2 w}{\partial z^2} + e_{15}\Delta\Phi + e_{33}\frac{\partial^2 \Phi}{\partial z^2} = 0,$$

$$\frac{1}{2}(e_{15} + e_{31})\frac{\partial}{\partial z}\left(\bar{\Lambda} U + \Lambda \bar{U}\right) + e_{15}\Delta w + e_{33}\frac{\partial^2 w}{\partial z^2} - \varepsilon_{11}\Delta\Phi - \varepsilon_{33}\frac{\partial^2 \Phi}{\partial z^2} = 0, \tag{2}$$

where, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\Lambda = \partial/\partial x + \mathrm{i}\partial/\partial y$, and the overbar indicates the complex conjugate value. The general solution to eqn (2) obtained by Ding et al. (1997a) is rewritten in the following form:

$$U = \Lambda\left(\sum_{i=1}^{3} F_i + \mathrm{i}F_4\right), \quad w = \sum_{i=1}^{3} \alpha_{i1}\frac{\partial F_i}{\partial z_i}, \quad \Phi = \sum_{i=1}^{3} \alpha_{i2}\frac{\partial F_i}{\partial z_i}, \tag{3}$$

where

$$\alpha_{i1} = \frac{c_{11}\varepsilon_{11} - m_3 s_i^2 + c_{44}\varepsilon_{33}s_i^4}{\left(m_1 - m_2 s_i^2\right)s_i}, \quad \alpha_{i2} = \frac{c_{11}e_{15} - m_4 s_i^2 + c_{44}e_{33}s_i^4}{\left(m_1 - m_2 s_i^2\right)s_i},$$

$$m_1 = \varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31}), \quad m_2 = \varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31}),$$

$$m_3 = c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2, \quad m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31}), \tag{4}$$

and $z_i = s_i z$, $s_4^2 = c_{66}/c_{44}$, and $s_i^2 (i = 1, 2, 3)$ are roots of the following algebraic equation:

$$as^6 - bs^4 + cs^2 - d = 0 \tag{5}$$

where,

$$a = c_{44}\left(e_{33}^2 + c_{33}\varepsilon_{33}\right), \quad b = c_{33}m_3 + \varepsilon_{33}\left[c_{44}^2 - (c_{13} + c_{44})^2\right] + e_{33}(2m_4 - c_{11}e_{33}),$$

$$c = c_{44}m_3 + \varepsilon_{11}\left[c_{11}c_{33} - (c_{13} + c_{44})^2\right] + e_{15}(2m_4 - c_{44}e_{15}), \quad d = c_{11}\left(e_{15}^2 + c_{44}\varepsilon_{11}\right).$$

It is noted here that the general solution given in eqn (3) is only valid for distinct s_i^2, while different forms should be adopted for other cases, see Appendix A.

Moreover, $F_i(z)$ satisfies the following quasi harmonic equations, respectively,

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2}\right)F_i = 0, \quad (i = 1, 2, 3, 4). \tag{6}$$

From eqns (1) and (3), the following expressions for stresses and electric displacements are derived:

$$\sigma_1 = 2\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}\right]F_i = 2\Delta\sum_{i=1}^{3}\left[(c_{11} - c_{66}) - c_{13}s_i\alpha_{i1} - e_{31}s_i\alpha_{i2}\right]F_i,$$

$$\sigma_2 = 2c_{66}\Lambda^2(F_1 + F_2 + F_3 + \mathrm{i}F_4), \quad \sigma_z = \sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\gamma_{1i}F_i = -\Delta\sum_{i=1}^{3}\gamma_{1i}F_i,$$

$$\tau_z = \Lambda\left\{\sum_{i=1}^{3}\left[c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2}\right]\frac{\partial}{\partial z_i}F_i + \mathrm{i}s_4c_{44}\frac{\partial}{\partial z_4}F_4\right\},$$

$$D = \Lambda\left\{\sum_{i=1}^{3}\left[e_{15}(s_i + \alpha_{i1}) - \varepsilon_{11}\alpha_{i2}\right]\frac{\partial}{\partial z_i}F_i + \mathrm{i}s_4e_{15}\frac{\partial}{\partial z_4}F_4\right\}, \quad D_z = \sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\gamma_{2i}F_i = -\Delta\sum_{i=1}^{3}\gamma_{2i}F_i, \tag{7}$$

where, $\sigma_1 = \sigma_x + \sigma_y$, $\sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy}$, $\tau_z = \tau_{xz} + \mathrm{i}\tau_{yz}$ and $D = D_x + \mathrm{i}D_y$, and,

$$\gamma_{1i} = -c_{13} + c_{33}s_i\alpha_{i1} + e_{33}s_i\alpha_{i2}, \; \gamma_{2i} = -e_{31} + e_{33}s_i\alpha_{i1} - \varepsilon_{33}s_i\alpha_{i2}. \tag{8}$$

3. The potential theory method

It is firstly considered that a smooth rigid punch with arbitrary end shape S is pressed against a transversely isotropic piezoelectric half-space $z \geqslant 0$ by a normal force P. The problem can be solved as follows: find the solution to the set of differential equations, eqn (2), for a half-space $z \geqslant 0$, subject to the mixed boundary conditions on the plane $z = 0$:

$$w = \omega(x, y, 0), \quad \Phi = \varphi(x, y, 0), \quad \text{for } (x, y) \in S;$$

$$\sigma_z = D_z = 0, \quad \text{for } (x, y) \notin S;$$

$$\tau_z = 0, \quad \text{for } -\infty < (x, y) < \infty \tag{9}$$

As usual (Fan et al., 1996), the displacement and electric potential are prescribed in the contact region as ω and φ, respectively. For the sake of practical convenience, the punch can be grounded and the electric potential will be zero. These conditions can be satisfied by a representation in terms of two harmonic functions G and H, i.e.

$$F_i(z) = c_i G(z_i) + d_i H(z_i), \quad (i = 1, 2, 3); \quad F_4(z) = 0 \tag{10}$$

where, to satisfy the third condition in eqn (9), one may set

$$\sum_{i=1}^{3} c_i \left[c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2} \right] = 0, \quad \sum_{i=1}^{3} d_i \left[c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2} \right] = 0 \tag{11}$$

and the two functions G and H are given as:

$$G(\rho, \phi, z) = \iint_s \ln\left[R(M, N) + z \right] \sigma_0(N)\, \mathrm{d}S, \quad H(\rho, \phi, z) = \iint_s \ln\left[R(M, N) + z \right] D_0(N)\, \mathrm{d}S, \tag{12}$$

where $\sigma_0(N)$ and $D_0(N)$ stand for values of σ_z and D_z at point $N(r, \psi, 0)$ respectively, $R(M, N)$ is the distance between the points $M(\rho, \phi, z)$ and $N(r, \psi, 0)$, and the integration is taken over the contact region S. Hereafter, the cylindrical coordinates (ρ, ϕ, z) are alternatively used for the sake of convenience. In contrast to pure elasticity, here a new potential H has been introduced to include the effect of electric field in piezoelectric materials. Making use of the property of the potential of a simple layer, it is known that the second condition in eqn (9) is already identically satisfied while inside the contact region S, one has

$$\left. \frac{\partial^2 G}{\partial z^2} \right|_{z=0} = -2\pi\sigma_0, \quad \left. \frac{\partial^2 H}{\partial z^2} \right|_{z=0} = -2\pi D_0. \tag{13}$$

The following relations then can be obtained from eqns (7), (10) and (13):

$$\sum_{i=1}^{3} c_i \gamma_{1i} = -\frac{1}{2\pi}, \quad \sum_{i=1}^{3} d_i \gamma_{1i} = 0, \quad \sum_{i=1}^{3} c_i \gamma_{2i} = 0, \quad \sum_{i=1}^{3} d_i \gamma_{2i} = -\frac{1}{2\pi}. \tag{14}$$

Since $c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2} = \gamma_{1i} s_i$, one can solve c_i and d_i from eqns (11) and (14) as follows:

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} \gamma_{11}s_i & \gamma_{12}s_2 & \gamma_{13}s_3 \\ \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} \gamma_{11}s_1 & \gamma_{12}s_2 & \gamma_{13}s_3 \\ \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{15}$$

Combining eqns (3) and (10) with eqn (12), the expressions of displacements and electric potential can be derived. Taking consideration of the first condition in eqn (9), the following integral equations are obtained:

$$\omega(N_0) = g_1\iint_s \frac{\sigma_0(N)}{R(N_0, N)}\mathrm{d}S + g_2\iint_s \frac{D_0(N)}{R(N_0, N)}\mathrm{d}S,$$

$$\varphi(N_0) = g_3\iint_s \frac{\sigma_0(N)}{R(N_0, N)}\mathrm{d}S + g_4\iint_s \frac{D_0(N)}{R(N_0, N)}\mathrm{d}S, \tag{16}$$

where, as above, $R(N_0, N)$ represents the distance between two points N_0 and N, and both $N_0, N \in S$. The constants g_i, $(i = 1, 2, 3, 4)$ are defined as follows:

$$g_1 = \sum_{i=1}^{3} c_i\alpha_{i1}, \quad g_2 = \sum_{i=1}^{3} d_i\alpha_{i1}, \quad g_3 = \sum_{i=1}^{3} c_i\alpha_{i2}, \quad g_4 = \sum_{i=1}^{3} d_i\alpha_{i2}. \tag{17}$$

From eqn (16), it is obtained that

$$g_4\omega(N_0) - g_2\varphi(N_0) = A\iint_s \frac{\sigma_0(N)}{R(N_0, N)}\,\mathrm{d}S, \tag{18}$$

$$g_1\varphi(N_0) - g_3\omega(N_0) = A\iint_s \frac{D_0(N)}{R(N_0, N)}\,\mathrm{d}S, \tag{19}$$

where $A = g_1g_4 - g_2g_3$. Thus, for the contact problem of arbitrary end shaped punch, two identical form integral equations have been set up, which can be solved using general numerical methods. However, for a circular punch, an explicit solution can be obtained by using the results of Fabrikant (1989). Moreover, for a flat circular punch maintained at a constant electric potential and loaded centrally by a concentrated force, a closed-form exact solution, which is expressed by elementary functions, can be obtained. This will be shown in the next section.

4. The circular punch

In the case that the punch is circular, by utilizing the results presented in Fabrikant (1989), the following expressions can be similarly obtained:

$$\frac{\partial G}{\partial z} = \frac{1}{\pi^2 A}\int_0^{2\pi}\int_0^a\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]\frac{z}{R_0^3}\left[g_4\omega(\rho_0, \phi_0) - g_2\varphi(\rho_0, \phi_0)\right]\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0,$$

$$\frac{\partial H}{\partial z} = \frac{1}{\pi^2 A}\int_0^{2\pi}\int_0^a\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]\frac{z}{R_0^3}\left[g_1\varphi(\rho_0, \phi_0) - g_3\omega(\rho_0, \phi_0)\right]\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \tag{20}$$

where a is the radius of the punch and,

$$R_0 = \left[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2\right]^{1/2}, \quad h = \left(a^2 - l_1^2\right)^{1/2}\left(a^2 - \rho_0^2\right)^{1/2}/a \tag{21}$$

and $l_1 = \frac{1}{2}\{[(\rho + a)^2 + z^2]^{1/2} - [(\rho - a)^2 + z^2]^{1/2}\}$.

Integrating eqn (20) gives

$$G(\rho, \phi, z) = \frac{1}{\pi^2 A}\int_0^{2\pi}\int_0^a K(\rho, \phi, z; \rho_0, \phi_0)\left[g_4\omega(\rho_0, \phi_0) - g_2\varphi(\rho_0, \phi_0)\right]\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0,$$

$$H(\rho, \phi, z) = \frac{1}{\pi^2 A}\int_0^{2\pi}\int_0^a K(\rho, \phi, z; \rho_0, \phi_0)\left[g_1\varphi(\rho_0, \phi_0) - g_3\omega(\rho_0, \phi_0)\right]\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \tag{22}$$

where the function K reads

$$K(\rho, \phi, z; \rho_0, \phi_0) = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{\left(a^2 - \rho_0^2\right)^{1/2}}\left\{\ln\left[\frac{a + \left(a^2 - l_1^2\right)^{1/2}}{l_1}\right]\right.$$
$$\left. + \frac{1}{(\zeta - 1)^{1/2}}\tan^{-1}\left[\frac{\left(a^2 - l_1^2\right)^{1/2}}{a(\zeta - 1)^{1/2}}\right] + \frac{1}{\left(\bar{\zeta} - 1\right)^{1/2}}\tan^{-1}\left[\frac{\left(a^2 - l_1^2\right)^{1/2}}{a\left(\bar{\zeta} - 1\right)^{1/2}}\right]\right\}, \tag{23}$$

where $\zeta = \mathrm{e}^{\mathrm{i}(\phi - \phi_0)}\rho/\rho_0$. Some derivatives of the function K (Green's functions related to elastoelectric field of displacements, stresses, electric displacements and electric potential) can be obtained in terms of elementary functions from eqn (23) by differentiation; these have been calculated by Fabrikant (1989) and are listed in Appendix B.

Now it is further assumed that the circular punch is flat ended, maintained at a constant electric potential and loaded centrally by a concentrated force. Under this consideration, it is known that both the electric potential φ and the punch settlement ω are constant inside the contact region. In this particular case, the following solutions to eqns (18) and (19) are obtained:

$$\sigma_0(\rho, \phi) = \frac{\beta_1}{\pi^2 A\left(a^2 - \rho^2\right)^{1/2}}, \quad D_0(\rho, \phi) = \frac{\beta_2}{\pi^2 A\left(a^2 - \rho^2\right)^{1/2}}, \tag{24}$$

where $\beta_1 = g_4\omega - g_2\varphi$ and $\beta_2 = g_1\varphi - g_3\omega$ are also constants. Substituting eqn (24) into eqn (12) gives the expressions for G and H as follows, respectively,

$$G(\rho, \phi, z) = \frac{2\beta_1}{\pi A}\left\{z\sin^{-1}\left(\frac{a}{l_2}\right) - \left(a^2 - l_1^2\right)^{1/2} + a\ln\left[l_2 + \left(l_2^2 - \rho^2\right)^{1/2}\right]\right\}, \tag{25}$$

$$H(\rho, \phi, z) = \frac{2\beta_2}{\pi A}\left\{z\sin^{-1}\left(\frac{a}{l_2}\right) - \left(a^2 - l_1^2\right)^{1/2} + a\ln\left[l_2 + \left(l_2^2 - \rho^2\right)^{1/2}\right]\right\} \tag{26}$$

where $l_2 = \frac{1}{2}\{[(\rho + a)^2 + z^2]^{1/2} + [(\rho - a)^2 + z^2]^{1/2}\}$. Substitution from eqns (25) and (26) into eqn (10) and these in turn into eqn (7) give rise to the expressions for all elastoelectric field variables which are obviously expressed in terms of elementary functions. The expressions for σ_z and D_z are given in the following as examples,

$$\sigma_z = -\frac{2}{\pi A}\sum_{i=1}^{3}\gamma_{1i}(c_i\beta_1 + d_i\beta_2)\frac{(a^2 - l_{1i}^2)^{1/2}}{l_{2i}^2 - l_{1i}^2},$$

$$D_z = -\frac{2}{\pi A}\sum_{i=1}^{3}\gamma_{2i}(c_i\beta_1 + d_i\beta_2)\frac{(a^2 - l_{1i}^2)^{1/2}}{l_{2i}^2 - l_{1i}^2}, \tag{27}$$

where $l_{1;2i} = \frac{1}{2}\{[(\rho + a)^2 + z_i^2]^{1/2} \mp [(\rho - a)^2 + z_i^2]^{1/2}\}$. It is interesting now to give the stress and electric displacement intensity factors at the edge of the punch, i.e. at $z = 0$ and $\rho = a$. Define the stress intensity factor and electric displacement intensity factor, respectively, as follows

$$k_\sigma = \lim_{\rho\to a}\left\{(a-\rho)^{1/2}\sigma_z|_{z=0}\right\}, \quad k_D = \lim_{\rho\to a}\left\{(a-\rho)^{1/2}D_z|_{z=0}\right\}. \tag{28}$$

Noticing eqn (14), eqns (27) and (28) yield

$$k_\sigma = \frac{\beta_1}{\pi^2 A(2a)^{1/2}}, \quad k_D = \frac{\beta_2}{\pi^2 A(2a)^{1/2}} \tag{29}$$

Now integrating the first equation in eqn (24) over the contact region gives the relation between the applied concentrated force P and the surface displacement ω as well as the electric potential φ,

$$P = -\frac{2\beta_1 a}{\pi A}. \tag{30}$$

Similarly, if we define the concentrated electric charge as Q, then integrating the second equation in eqn (24) gives,

$$Q = \frac{2\beta_2 a}{\pi A}. \tag{31}$$

Substitution of eqns (30) and (31) into eqn (29) yields,

$$k_\sigma = -\frac{P}{\pi(2a)^{3/2}}, \quad k_D = \frac{Q}{\pi(2a)^{3/2}}. \tag{32}$$

It is seen that the stress intensity factor has identically the same form as that for pure elasticity (the corresponding expression given in Fabrikant (1989) contains obvious printing errors). In other words, the coupling effect of piezoelectric material has no effect on the stress intensity factor. As regards to the electric displacement intensity factor, the similar conclusion can be reached.

5. Conclusion

The potential theory method has been generalized in the paper to analyze the piezoelastic contact problem of a punch pressed against a piezoelectric half-space. A new potential is introduced to take account of the effect of electric field. For the particular case that a flat centrally loaded circular punch is simultaneously maintained at a constant electric potential, an exact solution that is expressed in terms of elementary functions is then obtained. The corresponding stress and electric displacement intensity factors are defined in a usual manner and expressions of simple form are obtained. It is found that the stress intensity factor is independent of the material constants and is identically the same as that for pure elasticity.

It is worth mentioning here again that the general solution will take other forms for equal eigenvalue cases (Ding et al., 1997a). The succeeding derivations are similar to what has been described, see Appendix A. However, as pointed out by Fabrikant (1989), one can also derive the corresponding results of equal eigenvalues directly from the ones of distinct eigenvalues, but utilizing the well-known L'Hospital rule.

In analogy with pure elasticity, further developments can be expected by utilizing the present method along with the delicate results of Fabrikant (1989).

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Appendix A

The general solutions for the cases of multiple roots of s_i $(i = 1, 2, 3)$ are also expressed in terms of quasi harmonic functions F_i $(i = 1, 2, 3, 4)$that satisfy eqn (6) (Ding et al., 1997a). They can be rewritten in the following complex forms:

Case (1): $s_1 \neq s_4 = s_3$

$$U = \Lambda(F_1 + F_2 + z_2 F_3 + \mathrm{i}F_4),$$

$$w = \alpha_{11}\frac{\partial F_1}{\partial z_1} + \alpha_{21}\frac{\partial F_2}{\partial z_2} + \alpha_{21} z_2 \frac{\partial F_3}{\partial z_2} + \alpha_{41} F_3,$$

$$\Phi = \alpha_{12}\frac{\partial F_1}{\partial z_1} + \alpha_{22}\frac{\partial F_2}{\partial z_2} + \alpha_{22} z_2 \frac{\partial F_3}{\partial z_2} + \alpha_{42} F_3, \tag{A1}$$

where

$$\alpha_{41} = \frac{2\left(2c_{44}\varepsilon_{33}s_2^2 - m_3\right)s_2 - \left(m_1 - 3m_2 s_2^2\right)\alpha_{21}}{m_1 - m_2 s_2^2}, \quad \alpha_{42} = \frac{2\left(2c_{44}e_{33}s_2^2 - m_4\right)s_2 - \left(m_1 - 3m_2 s_2^2\right)\alpha_{22}}{m_1 - m_2 s_2^2}.$$

Case (2): $s_1 = s_2 = s_3$

$$U = \Lambda\left(F_1 + z_1 F_2 + z_1^2 \frac{\partial F_3}{\partial z_1} + \mathrm{i}F_4\right),$$

$$w = \alpha_{11}\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \alpha_{41}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \alpha_{51}F_3,$$

$$\Phi = \alpha_{12}\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \alpha_{42}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \alpha_{52}F_3, \tag{A2}$$

where

$$\alpha_{51} = \frac{2\left[3m_2(\alpha_{11}+\alpha_{41})s_1^2 - m_1\alpha_{41} + \left(6c_{44}\varepsilon_{33}s_1^2 - m_3\right)s_1\right]}{m_1 - m_2 s_1^2},$$

$$\alpha_{52} = \frac{2\left[3m_2(\alpha_{12}+\alpha_{42})s_1^2 - m_1\alpha_{42} + \left(6c_{44}e_{33}s_1^2 - m_4\right)s_1\right]}{m_1 - m_2 s_1^2}.$$

For Case (1), we assume

$$F_i(z) = c_iG_1(z_i) + d_iH_1(z_i), \quad (i = 1, 2),$$

$$F_3(z) = c_3G_2(z_3) + d_3H_2(z_3), \quad F_4(z) = 0 \tag{A3}$$

where,

$$G_1(\rho,\phi,z) = \iint_s \ln\left[R(M,N) + z\right]\sigma_0(N)\,\mathrm{d}S, \quad H_1(\rho,\phi,z) = \iint_s \ln\left[R(M,N) + z\right]D_0(N)\,\mathrm{d}S, \tag{A4}$$

and

$$G_2(\rho,\phi,z) = \iint_s \frac{\sigma_0(N)}{R(M,N)}\,\mathrm{d}S, \quad H_2(\rho,\phi,z) = \iint_s \frac{D_0(N)}{R(M,N)}\,\mathrm{d}S. \tag{A5}$$

For Case (2), we assume

$$F_1(z) = c_1G_1(z_1) + d_1H_1(z_1),$$

$$F_i(z) = c_iG_2(z_i) + d_iH_2(z_i), \quad (i = 2, 3), \quad F_4(z) = 0 \tag{A6}$$

where G_i, H_i $(i = 1, 2)$ have been shown in eqns (A4) and (A5). The followed derivatives are omitted for the sake of simplicity. It is just mentioned here that, for both cases, the resulting integral equations have the same structure as eqn (16) except for the involved constants. Therefore, previous results in potential theory can be also used to obtain the corresponding solutions.

Appendix B

The following derivatives of the function K can be found in Fabrikant (1989):

$$\frac{\partial K}{\partial z} = \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right], \tag{B1}$$

$$\Lambda K = \frac{q}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{z^2}{h\bar{q}R_0^2} - \frac{1}{\left(a^2-\rho_0^2\right)^{1/2}\bar{q}\left(\bar{\zeta}-1\right)^{1/2}}\tan^{-1}\left[\frac{\left(a^2-l_1^2\right)^{1/2}}{a\left(\bar{\zeta}-1\right)^{1/2}}\right], \tag{B2}$$

$$\frac{\partial^2 K}{\partial z^2} = \frac{1}{R_0^3}\left(1 - \frac{3z^2}{R_0^2}\right)\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] + \frac{1}{h\left(R_0^2 + h^2\right)}\left[\frac{z^2}{R_0^2} - \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2}\right], \tag{B3}$$

$$\Lambda\frac{\partial K}{\partial z} = -\frac{3zq}{R_0^5}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] + \frac{z}{h\left(R_0^2 + h^2\right)}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{q}{R_0^2}\right], \tag{B4}$$

$$\begin{aligned}\Lambda^2 K = &-\frac{3q^2}{R_0^5}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{q^2 h}{R_0^4\left(R_0^2 + h^2\right)} + \frac{2z^2}{h\bar{q}^2 R_0^2}\left(2 - \frac{z^2}{R_0^2}\right)\\ &+ \frac{1}{\left(a^2 - \rho_0^2\right)^{1/2}}\left\{\frac{3}{\bar{q}^2\left(\bar{\zeta} - 1\right)^{1/2}}\tan^{-1}\left[\frac{\left(a^2 - l_1^2\right)^{1/2}}{a\left(\bar{\zeta} - 1\right)^{1/2}}\right] + \frac{a\left(a^2 - l_1^2\right)^{1/2}}{\bar{q}^2\left(a^2\bar{\zeta} - l_1^2\right)} - \frac{a\left(a^2 - l_1^2\right)^{1/2}\rho^2 e^{2i\phi}}{l_1^2\left(R_0^2 + h^2\right)\left(l_2^2 - l_1^2\right)}\right\},\end{aligned} \tag{B5}$$

where $q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}$.

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